

# Spectrum Generating on Twistor Bundle

Thomas Branson and Doojin Hong

February 2, 2008

## Abstract

We give explicit formulas for the intertwinors of all orders on the twistor bundle over  $S^1 \times S^{n-1}$  using spectrum generating technique introduced in [5].

## 1 Introduction

It was shown in [5] that one can construct intertwining operators of some representations without too much effort when eigenspaces occur with multiplicity one. On the differential form bundle over  $S^1 \times S^{n-1}$ , the double cover of the compactified Minkowski space, some  $K$ -type eigenspaces occur with multiplicity two. After some additional computation, Branson also showed spectral function for these operators.

Intertwinors on spinors like the Dirac operator have eigenspaces with multiplicity one over  $S^1 \times S^{n-1}$  and explicit spectral function was given in [7]. But on twistors, the eigenspaces of the intertwinors including Rarita Schwinger operator have multiplicity two on some  $K$ -type. In this paper, we present the spectral function for these operators.

We briefly review conformal covariance and intertwining relation (for more details, see [2], [5]).

Let  $M$  be an  $n$ -dimensional spin manifold. We enlarge the structure group  $\text{Spin}(n)$  to  $\text{Spin}(n) \times \mathbb{R}_+$  in conformal geometry.  $(V(\lambda), \lambda^r)$  are finite dimensional  $\text{Spin}(n) \times \mathbb{R}_+$  representations, where  $(V(\lambda), \lambda)$  are finite dimensional representations of  $\text{Spin}(n)$  and  $\lambda^r(h, \alpha) = \alpha^r \lambda(h)$  for  $h \in \text{Spin}(n)$  and  $\alpha \in \mathbb{R}_+$ . The corresponding associated vector bundles are  $\mathbb{V}(\lambda) = P_{\text{Spin}(n)} \times_{\lambda} V(\lambda)$  and  $\mathbb{V}^r(\lambda) = P_{\text{Spin}(n) \times \mathbb{R}_+} \times_{\lambda^r} V(\lambda)$  with structure groups  $\text{Spin}(n)$  and  $\text{Spin}(n) \times \mathbb{R}_+$ .  $r$  is called the conformal weight of  $\mathbb{V}^r$ . Tangent bundle  $TM$  carries conformal weight  $-1$  and cotangent bundle  $T^*M$  carries conformal weight  $+1$ . In general, if  $V$  is a subbundle of  $(TM)^{\otimes p} \otimes (T^*M)^{\otimes q} \otimes (\Sigma M)^{\otimes r} \otimes (\Sigma^* M)^{\otimes s}$ , then  $V$  carries conformal weight  $q - p$ , where  $\Sigma M$  is the contravariant spinor bundle.

A conformal covariant of bidegree  $(a, b)$  is a  $\text{Spin}(n) \times \mathbb{R}_+$ -equivariant differential operator  $D : \mathbb{V}^r(\lambda) \rightarrow \mathbb{V}^s(\sigma)$  which is a polynomial in the metric  $g$ , its inverse  $g^{-1}$ , the volume element  $E$ , and the fundamental tensor-spinor  $\gamma$  with a conformal covariance law

$$\omega \in C^\infty, \quad \bar{g} = e^{2\omega} g, \quad \bar{E} = e^{n\omega} E, \quad \bar{\gamma} = e^{-\omega} \gamma \Rightarrow \bar{D} = e^{-b\omega} D \mu(e^{a\omega}),$$

where  $\mu(e^{a\omega})$  is multiplication of  $e^{a\omega}$ .

Given a conformal covariant of bidegree  $(a, b)$ ,  $D : \mathbb{V}^r(\lambda) \rightarrow \mathbb{V}^s(\sigma)$ , we can assign new conformal weights to get  $D : \mathbb{V}^{r'}(\lambda) \rightarrow \mathbb{V}^{s'}(\sigma)$  whose bidegree is then  $(a - r' + r, b - s' + s)$ . Calling this  $D$  again is an abuse of notation. If  $r' = r + a$  and  $s' = s + b$ , then  $D : \mathbb{V}^{r+a}(\lambda) \rightarrow \mathbb{V}^{s+b}(\sigma)$  becomes conformally invariant and we call  $(a + r, b + r)$  the reduced conformal bidegree of  $D$ . To see how conformal covariants behave under a conformal transformation and a conformal vector field, we recall followings.

A diffeomorphism  $h : M \rightarrow M$  is called a conformal transformation if  $h \cdot g = e^{2\omega_h} g$ , where  $\cdot$  is the natural action of  $h$  on tensor fields. A conformal vector field is a vector field  $X$  with  $\mathcal{L}_X g = 2\omega_X g$  for some  $\omega_X \in C^\infty(M)$ . A conformal covariant  $D : \mathbb{V}^0(\lambda) \rightarrow \mathbb{V}^0(\sigma)$  of reduced bidegree  $(a, b)$  satisfies

$$D(e^{a\omega_h} h \cdot \varphi) = e^{b\omega_h} h \cdot (D(\varphi)) \quad \text{and} \quad D(\mathcal{L}_X + a\omega_X)\varphi = (\mathcal{L}_X + b\omega_X)D\varphi.$$

for all  $\varphi \in \Gamma(\mathbb{V}^0(\lambda))$ . Thus if  $D : \mathbb{V}^r(\lambda) \rightarrow \mathbb{V}^s(\sigma)$  of reduced bidegree  $(a, b)$ , then

$$D(\mathcal{L}_X + (a - r)\omega_X)\varphi = (\mathcal{L}_X + (b - s)\omega_X)D\varphi \quad (1.1)$$

for  $\varphi \in \Gamma(\mathbb{V}^r(\lambda))$  and  $D\varphi \in \Gamma(\mathbb{V}^s(\sigma))$ .

Note that conformal vector fields form a Lie algebra  $\mathfrak{c}(M, g)$  and give rise to the principal series representation

$$U_a^\lambda : \mathfrak{c}(M, g) \rightarrow \text{End}\Gamma(\mathbb{V}^0(\lambda)) \quad \text{by} \quad X \mapsto \mathcal{L}_X + a\omega_X.$$

So a conformal covariant  $D : \mathbb{V}^r(\lambda) \rightarrow \mathbb{V}^s(\sigma)$  of reduced bidegree  $(a, b)$  intertwines the principal series representation

$$DU_{a-r}^\lambda \varphi = U_{b-s}^\sigma D\varphi$$

for  $\varphi \in \Gamma(\mathbb{V}^r(\lambda))$  and  $D\varphi \in \Gamma(\mathbb{V}^s(\sigma))$ .

## 2 Spinors and Twistors

Let  $M = S^1 \times S^{n-1}$ ,  $n$  even, be a manifold endowed with the Lorentz metric  $-dt^2 + g_{S^{n-1}}$ .

To get a fundamental tensor-spinor  $\alpha$  for  $M$  from the corresponding object  $\gamma$  on  $S^{n-1}$ , let

$$\alpha^j = \begin{pmatrix} \gamma^j & 0 \\ 0 & -\gamma^j \end{pmatrix}, \quad j = 1, \dots, n-1,$$

and

$$\alpha^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since  $M$  is even-dimensional, there is a *chirality* operator  $\chi_M$ , equal to some complex unit times  $\alpha^0 \tilde{\chi}_S$ , where

$$\tilde{\chi}_S = \begin{pmatrix} \chi_S & 0 \\ 0 & -\chi_S \end{pmatrix},$$

$\chi_S$  being the chirality operator on  $S$ . The chirality operator is always normalized to have square 1; thus  $(\chi_S)^2$  and  $(\tilde{\chi}_S)^2$  are identity operators, and since  $\alpha^0\alpha^0 = 1$ , we have  $(\alpha^0\tilde{\chi}_S)^2 = -1$ . As a result, we may take

$$\chi_M = \pm\sqrt{-1}\alpha^0\tilde{\chi}_S.$$

A *spinor* on  $M$  can be viewed as a pair of time-dependent spinors on  $S^{n-1}$ , i.e.,  $\begin{pmatrix} \varphi \\ \psi \end{pmatrix}$ , where  $\varphi$  and  $\psi$  are  $t$ -dependent spinors on  $S^{n-1}$ . But by chirality consideration ([6]), we get  $\Xi = \pm 1$  spinors:

$$\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} \Xi\psi/\sqrt{-1} \\ \psi \end{pmatrix}.$$

Recall that *twistors* are spinor-one-forms  $\Phi_\lambda$  with  $\alpha^\lambda\Phi_\lambda = 0$ . Given a chirality  $\Xi$ , a twistor  $\Psi$  is determined by a  $t$ -dependent spinor-one-form  $\psi_j$  on  $S^{n-1}$  via

$$\Psi = dt \wedge \begin{pmatrix} \varphi_0 \\ \psi_0 \end{pmatrix} + \begin{pmatrix} \varphi_j \\ \psi_j \end{pmatrix},$$

where

$$\begin{aligned} \varphi_j &= -\Xi\sqrt{-1}\psi_j, \\ \psi_0 &= \Xi\sqrt{-1}\gamma^k\psi_k, \\ \varphi_0 &= \gamma^k\psi_k. \end{aligned}$$

Furthermore, by Hodge theoretic consideration ([6]), twistors on  $M$  can be decomposed into three pieces

$$\begin{aligned} &\begin{pmatrix} -(n-1)\theta & -\Xi\sqrt{-1}\gamma_i\theta \\ -(n-1)\Xi\sqrt{-1}\theta & \gamma_i\theta \end{pmatrix} + \begin{pmatrix} 0 & -\Xi\sqrt{-1}T_i\tau \\ 0 & T_i\tau \end{pmatrix} + \begin{pmatrix} 0 & -\Xi\sqrt{-1}\nabla^j\eta_{ji} \\ 0 & \nabla^j\eta_{ji} \end{pmatrix} \\ &=: \langle\theta\rangle + \{\tau\} + [\eta]. \end{aligned} \tag{2.2}$$

### 3 Intertwining relation on twistors

Consider the standard conformal vector field ([1, 9])

$$T := \cos\rho \sin t \partial_t + \cos t \sin\rho \partial_\rho.$$

Here  $\rho$  is the azimuthal angle on  $S^{n-1}$ . The conformal factor of  $T$  is

$$\varpi := \cos t \cos\rho.$$

Let  $A = A_{2r}$  be an intertwinor of order  $2r$ . The intertwining relation says ((1.1), [2, 3, 5])

$$A\left(\tilde{\mathcal{L}}_T + \left(\frac{n}{2} - r\right)\varpi\right) = \left(\tilde{\mathcal{L}}_T + \left(\frac{n}{2} + r\right)\varpi\right)A, \tag{3.3}$$

where  $\tilde{\mathcal{L}}_T$  is the *reduced Lie derivative*. On a tensor-spinor with  $\begin{pmatrix} p \\ q \end{pmatrix}$  tensor content, this is

$$\tilde{\mathcal{L}}_T = \mathcal{L}_T + (p - q)\varpi.$$

So here (with only 1-form content), it is  $\mathcal{L}_T - \varpi$ . Note that we are using the convention where spinors do not have an internal weight; otherwise the spinor content would influence the reduction.

Since intertwinors change chirality, we want to consider an exchange operator

$$\begin{aligned} E : &= \alpha^0(\iota(\partial_t)\varepsilon(dt) - \varepsilon(dt)\iota(\partial_t)) \\ &= \alpha^0(1 - 2\varepsilon(dt)\iota(\partial_t)). \end{aligned}$$

It is immediate that  $E^2 = \text{Id}$ . Because of the  $\alpha^0$  factor,  $E$  reverses chirality. To see that  $E$  takes twistors to twistors, note that

$$\iota(\partial_t)\varepsilon(dt) - \varepsilon(dt)\iota(\partial_t) : \Phi_\lambda \mapsto \Phi_\lambda - 2\delta_\lambda^0 \Phi_0.$$

Thus

$$\begin{aligned} \alpha^\lambda(E\Phi)_\lambda &= \alpha^\lambda \alpha^0(\Phi_\lambda - 2\delta_\lambda^0 \Phi_0) \\ &= -2g^{\lambda 0}(\Phi_\lambda - 2\delta_\lambda^0 \Phi_0) + 2\alpha^0 \alpha^\lambda \delta_\lambda^0 \Phi_0 \\ &= \underbrace{-2\Phi_0}_{2\Phi_0} + 4 \underbrace{g^{00}}_{-1} \Phi_0 + 2 \underbrace{\alpha^0 \alpha^0}_1 \Phi_0 \\ &= 0, \end{aligned}$$

as desired.

We want to convert the relation (3.3) for  $EA$ . So we will eventually need  $\mathcal{L}_T E$ . We have:

$$\begin{aligned} \mathcal{L}_T E &= \mathcal{L}_T \{ \alpha(dt)(1 - 2\varepsilon(dt)\iota(\partial_t)) \} \\ &= \{ -\varpi \alpha(dt) + \alpha(d(Tt)) \} (1 - 2\varepsilon^0 \iota_0) \\ &\quad - 2\alpha^0 \{ \varepsilon(dt)\iota([T, \partial_t]) + \varepsilon(d(Tt)\iota(\partial_t)) \}. \end{aligned}$$

But

$$\begin{aligned} Tt &= \cos \rho \sin t, \\ d(Tt) &= -\sin \rho \sin t d\rho + \cos \rho \cos t dt, \\ [T, \partial_t] &= -\cos \rho \cos t \partial_t + \sin t \sin \rho \partial_\rho. \end{aligned}$$

This reduces the above to

$$\begin{aligned} \mathcal{L}_T E &= \sin t \alpha(d\omega)(1 - 2\varepsilon^0 \iota_0) - 2 \sin t \alpha^0(\varepsilon^0 \iota(Y) + \varepsilon(d\omega)\iota_0) \\ &= \sin t \sin \rho \{ -\alpha^1(1 - 2\varepsilon^0 \iota_0) - 2\alpha^0(\varepsilon^0 \iota_1 - \varepsilon^1 \iota_0) \}. \end{aligned} \tag{3.4}$$

By Kosmann ([8], eq(16)), the Lie and covariant derivatives on spinors are related by

$$\mathcal{L}_X - \nabla_X = -\frac{1}{4} \nabla_{[a} X_{b]} \gamma^a \gamma^b = -\frac{1}{8} (dX_b)_{ab} \gamma^a \gamma^b.$$

Note that

$$\begin{aligned} T_b &= -\cos \rho \sin t dt + \cos t \sin \rho d\rho, \\ dT_b &= 2 \sin \rho \sin t d\rho \wedge dt. \end{aligned}$$

and

$$d\varpi = -T_{b,R},$$

where  $\flat, \flat$  is the musical isomorphism in the “Riemannian” metric. According to the above,

$$\mathcal{L}_T - \nabla_T = -\frac{1}{2} \sin \rho \sin t \alpha^1 \alpha^0 \quad (3.5)$$

on spinors.

On a 1-form  $\eta$ ,

$$\langle (\mathcal{L}_T - \nabla_T) \eta, X \rangle = -\langle \eta, (\mathcal{L}_T - \nabla_T) X \rangle,$$

since  $\mathcal{L}_T - \nabla_T$  kills scalar functions. But by the symmetry of the pseudo-Riemannian connection,

$$[T, X] - \nabla_T X = -\nabla_X T.$$

We conclude that

$$(\mathcal{L}_T - \nabla_T) \eta = \langle \eta, \nabla T \rangle,$$

where in the last expression,  $\langle \cdot, \cdot \rangle$  is the pairing of a 1-form with the contravariant part of a  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ -tensor:

$$((\mathcal{L}_T - \nabla_T) \eta)_\lambda = \eta_\mu \nabla_\lambda T^\mu.$$

Combining this with what we derived above for spinors (3.5), for a spinor-1-form  $\Phi_\lambda$ , we have

$$((\mathcal{L}_T - \nabla_T) \Phi)_\lambda = \Phi_\mu \nabla_\lambda T^\mu - \frac{1}{2} \sin \rho \sin t \alpha^1 \alpha^0 \Phi_\lambda.$$

But  $\nabla T$  *a priori* has projections in 3 irreducible bundles,  $\text{TFS}^2$ ,  $\Lambda^0$ , and  $\Lambda^2$  (after using the musical isomorphisms). By conformality, the  $\text{TFS}^2$  part is gone. We expect a  $\Lambda^0$  part, essentially  $\varpi$ . We also found the  $\Lambda^2$  part above,

$$dT_b = 2 \sin \rho \sin t \, d\rho \wedge dt.$$

More precisely, tracking the normalizations,

$$(\nabla T_b)_{\lambda\mu} = (\nabla T_b)_{(\lambda\mu)} + (\nabla T_b)_{[\lambda\mu]} = (\varpi g + \frac{1}{2} dT_b)_{\lambda\mu}.$$

Now note that

$$\begin{aligned} \Phi_\mu \nabla_\lambda T^\mu &= (\nabla_\bullet T^\bullet \# \Phi)_\lambda \\ &= \varpi (g \# \Phi)_\lambda + \frac{1}{2} ((dT_b)_{\nu\mu} \varepsilon^\nu \iota^\mu \Phi)_\lambda \\ &= \varpi \Phi_\lambda + \frac{1}{2} (((dT_b)_{01} \varepsilon^0 \iota^1 + (dT_b)_{10} \varepsilon^1 \iota^0) \Phi)_\lambda \\ &= \varpi \Phi_\lambda + \frac{1}{2} ((-2 \sin \rho \sin t \varepsilon^0 \iota^1 + 2 \sin \rho \sin t \varepsilon^1 \iota^0) \Phi)_\lambda \\ &= \varpi \Phi_\lambda - \sin \rho \sin t ((\varepsilon^0 \iota^1 - \varepsilon^1 \iota^0) \Phi)_\lambda \\ &= \varpi \Phi_\lambda - \sin \rho \sin t ((\varepsilon^0 \iota_1 + \varepsilon^1 \iota_0) \Phi)_\lambda. \end{aligned}$$

As a result,

$$\begin{aligned} \mathcal{L}_T - \nabla_T &= \varpi - \sin \rho \sin t \left( \frac{1}{2} \alpha^1 \alpha^0 + \varepsilon^0 \iota_1 + \varepsilon^1 \iota_0 \right) \\ &=: \varpi - \sin \rho \sin t P \\ &=: \varpi - \mathcal{P}, \end{aligned}$$

and

$$\tilde{\mathcal{L}}_T - \nabla_T = -\mathcal{P}.$$

An explicit calculation using (3.4) gives

$$(\mathcal{L}_T E)E = -2\mathcal{P}.$$

Since  $E^2 = \text{Id}$ , we conclude that

$$\mathcal{L}_T E = -2\mathcal{P}E.$$

With the above, the intertwining relation for  $EA$  becomes

$$\begin{aligned} \left( \tilde{\mathcal{L}}_T + \left( \frac{n}{2} + r \right) \varpi \right) EA &= E \left( \tilde{\mathcal{L}}_T + \left( \frac{n}{2} + r \right) \varpi \right) A + (\mathcal{L}_T E)A \\ &= EA \left( \tilde{\mathcal{L}}_T + \left( \frac{n}{2} - r \right) \varpi \right) - 2\mathcal{P}EA, \end{aligned}$$

so that, with  $B = EA$ ,

$$B \left( \nabla_T + \left( \frac{n}{2} - r \right) \varpi - \mathcal{P} \right) = \left( \nabla_T + \left( \frac{n}{2} + r \right) \varpi + \mathcal{P} \right) B.$$

To see what  $P$  does let us define two convenient operations.

$$\psi_j \xrightarrow{\text{expa}} \begin{pmatrix} u & \Xi\psi_j/\sqrt{-1} \\ -\Xi u/\sqrt{-1} & \psi_j \end{pmatrix} \xrightarrow{\text{slot}} \psi_j,$$

where  $u = \gamma^k \psi_k$ .

Note that

$$\begin{aligned} \psi_j &\xrightarrow{\text{expa}} \begin{pmatrix} u & \Xi\psi_j/\sqrt{-1} \\ -\Xi u/\sqrt{-1} & \psi_j \end{pmatrix} \xrightarrow{\iota_0} \begin{pmatrix} u & \\ -\Xi u/\sqrt{-1} & \end{pmatrix} \\ &\xrightarrow{\varepsilon^1} \begin{pmatrix} 0 & \varepsilon^1 u \\ 0 & -\Xi \varepsilon^1 u/\sqrt{-1} \end{pmatrix} \xrightarrow{\text{slot}} -\Xi \varepsilon^1 u/\sqrt{-1}. \end{aligned}$$

As for the  $\varepsilon^0 \iota_1$  term, anything in the range of  $\varepsilon^0$  has a **slot** of 0.

Finally,

$$\begin{aligned} \psi_j &\xrightarrow{\text{expa}} \begin{pmatrix} u & \Xi\psi_j/\sqrt{-1} \\ -\Xi u/\sqrt{-1} & \psi_j \end{pmatrix} \xrightarrow{\alpha^0} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u & \Xi\psi_j/\sqrt{-1} \\ -\Xi u/\sqrt{-1} & \psi_j \end{pmatrix} \\ &= \begin{pmatrix} -\Xi u/\sqrt{-1} & \psi_j \\ u & \Xi\psi_j/\sqrt{-1} \end{pmatrix} \xrightarrow{\alpha^1} \begin{pmatrix} -\Xi \gamma^1 u/\sqrt{-1} & \gamma^1 \psi_j \\ -\gamma^1 u & -\Xi \gamma^1 \psi_j/\sqrt{-1} \end{pmatrix}. \end{aligned}$$

So

$$\begin{aligned} \text{slot } P \text{ expa} : \psi_j &\mapsto -\frac{1}{2} \Xi \gamma^1 \psi_j/\sqrt{-1} - \Xi(\varepsilon^1 u)_j/\sqrt{-1} \\ &= -\frac{\Xi}{\sqrt{-1}} \left( \frac{1}{2} \gamma^1 \psi_j + (\varepsilon^1 u)_j \right) = -\frac{\Xi}{\sqrt{-1}} \left( \frac{1}{2} \gamma^1 \psi_j + \delta_j^1 u \right). \end{aligned}$$

Up to a factor of a complex unit, **slot**  $P$  **expa** is

$$\frac{1}{2} \gamma^1 \psi_j + \delta_j^1 \gamma^k \psi_k.$$

We can also get this expression by successively taking the commutator of  $\varpi$  with  $\partial_t$  and

$$\text{slot } \mathcal{D} \text{ expa} : \psi_j \mapsto \frac{1}{2} \gamma^k \nabla_k \psi_j + \gamma^k \nabla_j \psi_k.$$

That is,

$$\mathcal{P} = \Xi \sqrt{-1} [\partial_t, [\mathcal{D}, \varpi]].$$

Recall that  $\mathcal{P} = \sin \rho \sin t P$ .

After some straightforward computation, we get the block matrix for  $\mathcal{D}$  relative to the decomposition  $\{(\theta), \{\tau\}, [\eta]\}$  (2.2) as follows.

$$\begin{pmatrix} \frac{n+1}{2(n-1)} J_\theta & \frac{n-2}{4} - \frac{n-2}{(n-1)^2} J_\tau^2 & 0 \\ -n & \frac{n-3}{2(n-1)} J_\tau & 0 \\ 0 & 0 & \frac{1}{2} L \end{pmatrix},$$

where  $J_\theta$  and  $J_\tau$  are the Dirac eigenvalues of  $\theta$  and  $\tau$  on  $S^{n-1}$ , respectively and  $L$  is the Rarita-Schwinger eigenvalue of  $[\eta]$  on  $S^{n-1}$ .

The spectrum generating relation takes the following form:

$$[N, \varpi] = 2 \left( \nabla_T + \frac{n}{2} \varpi \right),$$

where  $\nabla^{*,R} \nabla := N$  is the Riemannian Bochner Laplacian. Therefore the relation (3.3) becomes

$$B \left( \frac{1}{2} [N, \varpi] - r \varpi - \Xi \sqrt{-1} [\partial_t, [\mathcal{D}, \varpi]] \right) = \left( \frac{1}{2} [N, \varpi] + r \varpi + \Xi \sqrt{-1} [\partial_t, [\mathcal{D}, \varpi]] \right) B. \quad (3.6)$$

As explained in detail in ([3]), the recursive numerical spectral data come from the compressed relation of the above.

## 4 Projections into isotypic summands

Let us denote the  $K = \text{Spin}(2) \times \text{Spin}(n)$ -type with highest weight

$$(f) \otimes (j, \frac{1}{2} + q \frac{1}{2}, \dots, \frac{1}{2}, \frac{\varepsilon}{2}),$$

where  $j \in \frac{1}{2} + q + \mathbb{N}$ ,  $\varepsilon = \pm 1$ , and  $q = 0$  or  $1$ , by

$$\mathcal{V}_\Xi(f, j, \frac{1}{2} + q \frac{1}{2}, \dots, \frac{1}{2}, \frac{\varepsilon}{2}).$$

An  $\mathfrak{s}$ -map from such a  $K$ -type lands in the direct sum of neighboring  $K$ -types ([1]).

Consider a  $\Xi$  spinor  $\begin{pmatrix} \varphi \\ \psi \end{pmatrix}$ . Since  $\varphi = \Xi \psi / \sqrt{-1}$ , we have

$$\alpha^0 \begin{pmatrix} \bullet \\ \psi \end{pmatrix} = \begin{pmatrix} \bullet \\ \Xi \psi / \sqrt{-1} \end{pmatrix}.$$

Here  $\bullet$  denotes a top entry that is computable from the bottom entry, but whose value is not needed at the moment.

In addition,

$$\begin{aligned} \sin t \begin{pmatrix} \bullet \\ \psi \end{pmatrix} &= \begin{pmatrix} \bullet \\ \sin t \psi \end{pmatrix} = \begin{pmatrix} \bullet \\ -[\partial_t, \cos t] \psi \end{pmatrix}, \\ \text{Proj}_{f'} \sin t \begin{pmatrix} \bullet \\ \psi \end{pmatrix} &= \begin{pmatrix} \bullet \\ \frac{f'-f}{\sqrt{-1}} \cos t|_f^{f'} \psi \end{pmatrix}, \\ \sin \rho \alpha^1 \begin{pmatrix} \bullet \\ \psi \end{pmatrix} &= \begin{pmatrix} \bullet \\ -\sin \rho \gamma^1 \psi \end{pmatrix} = \begin{pmatrix} \bullet \\ [D, \cos \rho] \psi \end{pmatrix}, \\ \text{Proj}_b \sin \rho \alpha^1 \begin{pmatrix} \bullet \\ \psi \end{pmatrix} &= \begin{pmatrix} \bullet \\ -\text{Proj}_b \sin \rho \gamma^1 \psi \end{pmatrix} = \begin{pmatrix} \bullet \\ D|_a^b \cos \rho \psi \end{pmatrix}, \end{aligned}$$

where  $D = \gamma^i \nabla_i$  is the Dirac operator on  $S^{n-1}$ . Here  $a$  and  $b$  (resp.,  $f$  and  $f'$ ) are abbreviated labels for the  $\text{Spin}(n)$ -types (resp.,  $\text{Spin}(2)$ -types) in question.

Note also that the compressed relations of  $\varpi$  between Clifford range part, twistor range part, and divergence part look (2.2):

$$\begin{aligned} \varpi \begin{pmatrix} \langle \theta \rangle \\ 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} \langle |\varpi| \theta \rangle \\ 0 \\ 0 \end{pmatrix} \xrightarrow{\text{Proj}}: \begin{pmatrix} \langle \tilde{\theta} \rangle \\ 0 \\ 0 \end{pmatrix}, \\ \varpi \begin{pmatrix} 0 \\ \{\tau\} \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 \\ |\varpi| \{\tau\} \\ |\varpi| \{\tau\} \end{pmatrix} = \begin{pmatrix} 0 \\ C \{|\varpi| \tau\} \\ |\varpi| \{\tau\} \end{pmatrix} \xrightarrow{\text{Proj}}: \begin{pmatrix} 0 \\ C \{\tilde{\tau}\} \\ [\eta] \end{pmatrix}, \\ \varpi \begin{pmatrix} 0 \\ 0 \\ [\eta] \end{pmatrix} &= \begin{pmatrix} 0 \\ |\varpi| [\eta] \\ |\varpi| [\eta] \end{pmatrix} \xrightarrow{\text{Proj}}: \begin{pmatrix} 0 \\ \{\tilde{\tau}\} \\ [\tilde{\eta}] \end{pmatrix}, \end{aligned} \quad (4.7)$$

where  $C$  is a quantity we will compute in the following lemma.

Note that  $\varpi \langle \theta \rangle$  has only Clifford range pieces, since it is made of a spinor and fundamental tensor-spinor on  $S^{n-1}$ . On the other hand,  $\varpi \{\tau\}$  and  $\varpi [\eta]$  have no Clifford range pieces, since they are made of twistors on  $S^{n-1}$  (See [2, 3]).

**Lemma 4.1.** *Let  $\alpha = \mathcal{V}_\Xi(f; j, \frac{1}{2}, \dots, \frac{1}{2}, \frac{\varepsilon}{2})$  and  $\beta = \mathcal{V}_\Xi(f'; j', \frac{1}{2}, \dots, \frac{1}{2}, \frac{\varepsilon'}{2})$ ,  $\varepsilon = \pm 1$ . Then we have*

$$|\beta \varpi|_\alpha \{\tau\} = C_{ba} \{|\beta \varpi|_\alpha \tau\},$$

where

$$C_{ba} = \frac{1}{\lambda_b(T^*T)} \left( \frac{1}{2} J_b^2 + \frac{1}{2} J_a^2 - \frac{J_b J_a}{n-1} - \frac{n(n-1)}{4} \right),$$

$J_a$  (resp.,  $J_b$ ) is the Dirac eigenvalue on  $\mathcal{V}_\Xi(j, \frac{1}{2}, \dots, \frac{1}{2}, \frac{\varepsilon}{2})$  (resp.,  $\mathcal{V}_\Xi(j', \frac{1}{2}, \dots, \frac{1}{2}, \frac{\varepsilon'}{2})$ ), and  $\lambda_b(T^*T)$  is the eigenvalue of  $T^*T$  on  $\mathcal{V}_\Xi(j', \frac{1}{2}, \dots, \frac{1}{2}, \frac{\varepsilon'}{2})$  over  $S^{n-1}$ .

*Proof.* It suffices to show for the twistor operator  $T$  on  $S^{n-1}$  and  $\omega = \cos \rho$  that

$$|_b \omega|_a T \tau = C_{ba} \cdot T(|_b \omega|_a \tau).$$



Let  $D$  be the Dirac operator on  $S^{n-1}$ . Then

$$\begin{aligned} [D^2, \omega]\tau &= [\nabla^* \nabla, \omega]\tau \text{ by Bochner identity} \\ &= (\nabla^* \nabla \omega)\tau - 2\nabla^k \omega \nabla_k \tau = (n-1)\omega\tau + 2\sin \rho \nabla_1 \tau, \end{aligned}$$

Also

$$\begin{aligned} T^*(\omega T\tau) &= -\nabla^j(\omega \nabla_j \tau + \frac{1}{n-1} \omega \gamma_j D\tau) \\ &= \sin \rho \nabla_1 \tau + \omega \nabla^* \nabla \tau + \frac{1}{n-1} \sin \rho \gamma_1 D\tau - \frac{1}{n-1} \omega D^2 \tau \\ &= \frac{1}{2} ([D^2, \omega] - (n-1)\omega) \tau + \omega \left( D^2 - \frac{(n-1)(n-2)}{4} \right) \tau + \frac{1}{n-1} [\omega, D] D\tau \\ &\quad - \frac{1}{n-1} \omega D^2 \tau \text{ by the above and Bochner identity} \\ &= \frac{1}{2} D^2(\omega\tau) + \frac{1}{2} \omega D^2 \tau - \frac{1}{n-1} D(\omega D\tau) - \frac{n(n-1)}{4} \omega\tau. \end{aligned}$$

Therefore

$$\begin{aligned} |b\omega|_a T\tau &= T \left( \frac{1}{\lambda_b(T^*T)} T^*(|b\omega|_a T\tau) \right) \\ &= T \left( \frac{1}{\lambda_b(T^*T)} \left( \frac{1}{2} J_b^2 + \frac{1}{2} J_a^2 - \frac{1}{n-1} J_b J_a - \frac{n(n-1)}{4} \right) |b\omega|_a \tau \right). \end{aligned}$$

□

**Remark 1.** Eigenvalues of  $D$  and  $T^*T$  on  $S^{n-1}$  are known due to Branson ([4]).

With the above (4.7) at hand, we get

$$\begin{aligned} |_\beta [\mathcal{D}, \varpi]|_\alpha \langle \theta \rangle &= \begin{pmatrix} (\mathcal{D}_{11}^\beta - \mathcal{D}_{11}^\alpha) \langle \tilde{\theta} \rangle \\ (\mathcal{D}_{21}^\beta - C_{ba} \mathcal{D}_{21}^\alpha) \{ \tilde{\theta} \} \\ -\mathcal{D}_{21}^\alpha [\eta] \end{pmatrix}, \text{ where } \begin{cases} \langle \tilde{\theta} \rangle = |_\beta \varpi|_\alpha \langle \theta \rangle \\ [\eta] = |_\beta \varpi|_\alpha \{ \theta \} \end{cases}, \\ |_\beta [\mathcal{D}, \varpi]|_\alpha \{ \tau \} &= \begin{pmatrix} (C_{ba} \mathcal{D}_{12}^\beta - \mathcal{D}_{12}^\alpha) \langle \tilde{\tau} \rangle \\ C_{ba} (\mathcal{D}_{22}^\beta - \mathcal{D}_{22}^\alpha) \{ \tilde{\tau} \} \\ (\mathcal{D}_{33}^\beta - \mathcal{D}_{22}^\alpha) [\eta] \end{pmatrix}, \text{ where } \begin{cases} \{ \tilde{\tau} \} = |_\beta \varpi|_\alpha \{ \tau \} \\ [\eta] = |_\beta \varpi|_\alpha \{ \tau \} \end{cases}, \text{ and} \\ |_\beta [\mathcal{D}, \varpi]|_\alpha [\eta] &= \begin{pmatrix} \mathcal{D}_{12}^\beta \langle \tilde{\tau} \rangle \\ (\mathcal{D}_{22}^\beta - \mathcal{D}_{33}^\alpha) \{ \tilde{\tau} \} \\ (\mathcal{D}_{33}^\beta - \mathcal{D}_{33}^\alpha) [\tilde{\eta}] \end{pmatrix}, \text{ where } \begin{cases} \{ \tilde{\tau} \} = |_\beta \varpi|_\alpha [\eta] \\ [\tilde{\eta}] = |_\beta \varpi|_\alpha [\eta] \end{cases}. \end{aligned} \tag{4.8}$$

Here we use subscripts to refer to the specific entries of the  $\mathcal{D}$  and superscripts to indicate where these entries are computed.

Let us now consider the compressed relation of (3.6) between neighboring  $K$ -types.

**Case 1: Multiplicity  $2 \leftrightarrow 1$**

$$\alpha = \mathcal{V}_\Xi(f; j, \frac{1}{2}, \dots, \frac{1}{2}, \frac{\varepsilon}{2}) \leftrightarrow \beta = \mathcal{V}_\Xi(f'; j, \frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2}, \frac{\varepsilon}{2}).$$

Note that the operator  $B$  in block form looks

$$B = \begin{pmatrix} B_{11} & B_{12} & 0 \\ B_{21} & B_{22} & 0 \\ 0 & 0 & B_{33} \end{pmatrix}.$$

With

$$|_{\alpha}N|_{\beta} = f^2 - f'^2 - (n-2) \text{ and } |_{\beta}N|_{\alpha} = -|_{\alpha}N|_{\beta}$$

and (4.8), we get  $\alpha \rightarrow \beta$  transition quantities

$$\begin{aligned} \beta \rightarrow \alpha : \quad & \begin{pmatrix} B_{11}^{\alpha} & B_{12}^{\alpha} \\ B_{21}^{\alpha} & B_{22}^{\alpha} \end{pmatrix} \begin{pmatrix} A_1 \\ E^- \end{pmatrix} = B_{33}^{\beta} \begin{pmatrix} -A_1 \\ E^+ \end{pmatrix} \text{ and} \\ \alpha \rightarrow \beta : \quad & \begin{pmatrix} A_2 & -E^- \end{pmatrix} \begin{pmatrix} B_{11}^{\alpha} & B_{12}^{\alpha} \\ B_{21}^{\alpha} & B_{22}^{\alpha} \end{pmatrix} = B_{33}^{\beta} \begin{pmatrix} -A_2 & -E^+ \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} A_1 &:= \Xi(f - f')\mathcal{D}_{12}^{\alpha}, \\ A_2 &:= -\Xi(f - f')\mathcal{D}_{21}^{\alpha}, \\ E^- &:= \frac{1}{2}(f^2 - f'^2) - \frac{n-2}{2} - r + \Xi(f - f')(\mathcal{D}_{22}^{\alpha} - \mathcal{D}_{33}^{\beta}), \\ E^+ &:= \frac{1}{2}(f^2 - f'^2) - \frac{n-2}{2} + r - \Xi(f - f')(\mathcal{D}_{22}^{\alpha} - \mathcal{D}_{33}^{\beta}). \end{aligned}$$

In particular, we can write all  $2 \times 2$  entries of  $B^{\alpha}$  in terms of  $B_{21}^{\alpha}$  and  $B_{33}^{\beta}$ :

$$\begin{aligned} B_{11}^{\alpha} &= (E^- B_{21}^{\alpha} - A_2 B_{33}^{\beta})/A_2, \\ B_{12}^{\alpha} &= -A_1 B_{21}^{\alpha}/A_2, \text{ and} \\ B_{22}^{\alpha} &= (-A_1 B_{21}^{\alpha} + E^+ B_{33}^{\beta})/E^-. \end{aligned} \tag{4.9}$$

Thus if we can express  $B_{21}^{\alpha}$  in terms of  $B_{33}^{\beta}$ , we can completely determine all entries in the  $2 \times 2$  block.

## Case 2: Multiplicity 2 $\leftrightarrow$ 2

$$\alpha = \mathcal{V}_{\Xi}(f; j, \frac{1}{2}, \dots, \frac{1}{2}, \frac{\varepsilon}{2}) \rightarrow \beta = \mathcal{V}_{\Xi}(f'; j', \frac{1}{2}, \dots, \frac{1}{2}, \frac{\varepsilon'}{2}).$$

Here we have

$$|_{\beta}N|_{\alpha} = f'^2 - f^2 + J_b^2 - J_a^2.$$

So using (4.8), we get the transition quantities

$$\begin{pmatrix} B_{11}^{\beta} & B_{12}^{\beta} \\ B_{21}^{\beta} & B_{22}^{\beta} \end{pmatrix} \begin{pmatrix} F_1^- & G_2 \\ G_1 & C_{ba}F_2^- \end{pmatrix} = \begin{pmatrix} F_1^+ & -G_2 \\ -G_1 & C_{ba}F_2^+ \end{pmatrix} \begin{pmatrix} B_{11}^{\alpha} & B_{12}^{\alpha} \\ B_{21}^{\alpha} & B_{22}^{\alpha} \end{pmatrix}, \tag{4.10}$$

where

$$\begin{aligned} F_1^- &:= \frac{1}{2}(f'^2 - f^2) + \frac{1}{2}(J_b^2 - J_a^2) - r + \Xi(f' - f)(\mathcal{D}_{11}^{\beta} - \mathcal{D}_{11}^{\alpha}), \\ F_1^+ &:= \frac{1}{2}(f'^2 - f^2) + \frac{1}{2}(J_b^2 - J_a^2) + r - \Xi(f' - f)(\mathcal{D}_{11}^{\beta} - \mathcal{D}_{11}^{\alpha}), \\ F_2^- &:= \frac{1}{2}(f'^2 - f^2) + \frac{1}{2}(J_b^2 - J_a^2) - r + \Xi(f' - f)(\mathcal{D}_{22}^{\beta} - \mathcal{D}_{22}^{\alpha}), \\ F_2^+ &:= \frac{1}{2}(f'^2 - f^2) + \frac{1}{2}(J_b^2 - J_a^2) + r - \Xi(f' - f)(\mathcal{D}_{22}^{\beta} - \mathcal{D}_{22}^{\alpha}), \\ G_1 &:= \Xi(f' - f)(\mathcal{D}_{21}^{\beta} - C_{ba}\mathcal{D}_{21}^{\alpha}), \text{ and} \\ G_2 &:= \Xi(f' - f)(C_{ba}\mathcal{D}_{12}^{\beta} - \mathcal{D}_{12}^{\alpha}). \end{aligned}$$

Therefore we get determinant quotients of  $B$  on multiplicity 2 part.

Note the following diagram of neighboring multiplicity 2 isotypic summands centered

at  $\mathcal{V}_\Xi(f; j, \frac{1}{2}, \dots, \frac{1}{2}, \frac{\varepsilon}{2})$ :

$$\begin{array}{ccc}
 \mathcal{V}_\Xi(f-1; j+1, \frac{1}{2}, \dots, \frac{1}{2}, \frac{\varepsilon}{2}) & & \mathcal{V}_\Xi(f+1; j+1, \frac{1}{2}, \dots, \frac{1}{2}, \frac{\varepsilon}{2}) \\
 & \swarrow \quad \searrow & \\
 \mathcal{V}_\Xi(f-1; j, \frac{1}{2}, \dots, \frac{1}{2}, -\frac{\varepsilon}{2}) & \leftarrow \bullet \rightarrow & \mathcal{V}_\Xi(f+1; j, \frac{1}{2}, \dots, \frac{1}{2}, -\frac{\varepsilon}{2}) \\
 & \swarrow \quad \searrow & \\
 \mathcal{V}_\Xi(f-1; j-1, \frac{1}{2}, \dots, \frac{1}{2}, \frac{\varepsilon}{2}) & & \mathcal{V}_\Xi(f+1; j-1, \frac{1}{2}, \dots, \frac{1}{2}, \frac{\varepsilon}{2}).
 \end{array}$$

The determinant quotients corresponding to the above diagram are:

$$\left( \begin{array}{cc}
 \frac{(-f+J+1-\Xi+r+\frac{\varepsilon}{2}\Xi)(-f+J+1+\Xi+r+\frac{\varepsilon}{2}\Xi)}{(-f+J+1-\Xi-r-\frac{\varepsilon}{2}\Xi)(-f+J+1+\Xi-r-\frac{\varepsilon}{2}\Xi)} & \frac{(f+J+1-\Xi+r-\frac{\varepsilon}{2}\Xi)(f+J+1+\Xi+r-\frac{\varepsilon}{2}\Xi)}{(f+J+1-\Xi-r+\frac{\varepsilon}{2}\Xi)(f+J+1+\Xi-r+\frac{\varepsilon}{2}\Xi)} \\
 \frac{(-f+\frac{1}{2}-\Xi+r-\varepsilon\Xi J)(-f+\frac{1}{2}+\Xi+r-\varepsilon\Xi J)}{(-f+\frac{1}{2}-\Xi-r+\varepsilon\Xi J)(-f+\frac{1}{2}+\Xi-r+\varepsilon\Xi J)} & \frac{(f+\frac{1}{2}-\Xi+r+\varepsilon\Xi J)(f+\frac{1}{2}+\Xi+r+\varepsilon\Xi J)}{(f+\frac{1}{2}-\Xi-r-\varepsilon\Xi J)(f+\frac{1}{2}+\Xi-r-\varepsilon\Xi J)} \\
 \frac{(-f-J+1-\Xi+r-\frac{\varepsilon}{2}\Xi)(-f-J+1+\Xi+r-\frac{\varepsilon}{2}\Xi)}{(-f-J+1-\Xi-r+\frac{\varepsilon}{2}\Xi)(-f-J+1+\Xi-r+\frac{\varepsilon}{2}\Xi)} & \frac{(f-J+1-\Xi+r+\frac{\varepsilon}{2}\Xi)(f-J+1+\Xi+r+\frac{\varepsilon}{2}\Xi)}{(f-J+1-\Xi-r-\frac{\varepsilon}{2}\Xi)(f-J+1+\Xi-r-\frac{\varepsilon}{2}\Xi)}
 \end{array} \right), \quad (4.11)$$

where  $J = \varepsilon J_a$ .

And these data can be put into the following gamma function expression:

$$\begin{aligned}
 & \frac{1}{4} \bullet \frac{\Gamma(\frac{1}{2}(f+J+r-\frac{\varepsilon}{2}\Xi)) \Gamma(\frac{1}{2}(-f+J+r+\frac{\varepsilon}{2}\Xi))}{\Gamma(\frac{1}{2}(f+J-r+\frac{\varepsilon}{2}\Xi)) \Gamma(\frac{1}{2}(-f+J-r-\frac{\varepsilon}{2}\Xi))} \\
 & \bullet \frac{\Gamma(\frac{1}{2}(f+J+2+r-\frac{\varepsilon}{2}\Xi)) \Gamma(\frac{1}{2}(-f+J+2+r+\frac{\varepsilon}{2}\Xi))}{\Gamma(\frac{1}{2}(f+J+2-r+\frac{\varepsilon}{2}\Xi)) \Gamma(\frac{1}{2}(-f+J+2-r-\frac{\varepsilon}{2}\Xi))}.
 \end{aligned}$$

### Case 3: Multiplicity $1 \leftrightarrow 1$

$$\alpha = \mathcal{V}_\Xi(f; j, \frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2}, \frac{\varepsilon}{2}) \leftarrow \beta = \mathcal{V}_\Xi(f'; j', \frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2}, \frac{\varepsilon'}{2}).$$

Again we have

$$|_\alpha N|_\beta = f^2 - f'^2 + J_a^2 - J_b^2.$$

And the transition quantities are

$$B_{33}^\alpha P^- = P^+ B_{33}^\beta, \quad (4.12)$$

where

$$\begin{aligned}
 P^- &:= \frac{1}{2}(f^2 - f'^2) + \frac{1}{2}(J_a^2 - J_b^2) - r + \Xi(f - f')(\mathcal{D}_{33}^\alpha - \mathcal{D}_{33}^\beta) \text{ and} \\
 P^+ &:= \frac{1}{2}(f^2 - f'^2) + \frac{1}{2}(J_a^2 - J_b^2) + r - \Xi(f - f')(\mathcal{D}_{33}^\alpha - \mathcal{D}_{33}^\beta).
 \end{aligned}$$

The diagram of neighboring multiplicity 1 isotypic summands centered at

$$\mathcal{V}_\Xi(f; j, \frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2}, \frac{\varepsilon}{2})$$

looks:

$$\begin{array}{ccc}
\mathcal{V}_{\Xi}(f-1; j+1, \frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2}, \frac{\varepsilon}{2}) & & \mathcal{V}_{\Xi}(f+1; j+1, \frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2}, \frac{\varepsilon}{2}) \\
& \swarrow \quad \nearrow & \\
\mathcal{V}_{\Xi}(f-1; j, \frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2}, -\frac{\varepsilon}{2}) & \leftarrow \bullet \rightarrow & \mathcal{V}_{\Xi}(f+1; j, \frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2}, -\frac{\varepsilon}{2}) \\
& \swarrow \quad \searrow & \\
\mathcal{V}_{\Xi}(f-1; j-1, \frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2}, \frac{\varepsilon}{2}) & & \mathcal{V}_{\Xi}(f+1; j-1, \frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2}, \frac{\varepsilon}{2}).
\end{array}$$

And the eigenvalue quotients are:

$$\begin{pmatrix} \frac{-f+J+1+r+\frac{\varepsilon}{2}\Xi}{-f+J+1-r-\frac{\varepsilon}{2}\Xi} & \frac{f+J+1+r-\frac{\varepsilon}{2}\Xi}{f+J+1-r+\frac{\varepsilon}{2}\Xi} \\ \frac{-f+\frac{1}{2}+r-\varepsilon\Xi J}{-f+\frac{1}{2}-r+\varepsilon\Xi J} & \frac{f+\frac{1}{2}+r+\varepsilon\Xi J}{f+\frac{1}{2}-r-\varepsilon\Xi J} \\ \frac{-f-J+1+r-\frac{\varepsilon}{2}\Xi}{-f-J+1-r+\frac{\varepsilon}{2}\Xi} & \frac{f-J+1+r+\frac{\varepsilon}{2}\Xi}{f-J+1-r-\frac{\varepsilon}{2}\Xi} \end{pmatrix},$$

where  $J = \varepsilon J_a$ .

Thus, following the normalization on the multiplicity 2 part, we get the spectral function on the multiplicity 1 part:

$$Z(r; f, J, \Xi\varepsilon) = \frac{\varepsilon}{2\Xi} \frac{\Gamma\left(\frac{1}{2}(f+J+1+r-\frac{\varepsilon}{2}\Xi)\right) \Gamma\left(\frac{1}{2}(-f+J+1+r+\frac{\varepsilon}{2}\Xi)\right)}{\Gamma\left(\frac{1}{2}(f+J+1-r+\frac{\varepsilon}{2}\Xi)\right) \Gamma\left(\frac{1}{2}(-f+J+1-r-\frac{\varepsilon}{2}\Xi)\right)}. \quad (4.13)$$

In particular,

$$Z\left(\frac{1}{2}, f, J, \Xi\varepsilon\right) = -\frac{1}{4}(f - \Xi\varepsilon J) = \frac{1}{4}\sqrt{-1} \operatorname{eig}(E\mathcal{R}; f, J, \Xi\varepsilon),$$

where  $E\mathcal{R}$  is the exchanged Rarita-Schwinger operator.

## 5 Interface between multiplicity 1 and 2 parts

Consider the following diagram:

$$\begin{array}{ccc}
\alpha_1 = \mathcal{V}_{\Xi}(f; j, \frac{1}{2}, \dots, \frac{1}{2}, \frac{\varepsilon}{2}) & \rightarrow & \alpha_2 = \mathcal{V}_{\Xi}(f+1; j+1, \frac{1}{2}, \dots, \frac{1}{2}, \frac{\varepsilon}{2}) \\
\downarrow & & \downarrow \\
\beta_1 = \mathcal{V}_{\Xi}(f+1; j, \frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2}, \frac{\varepsilon}{2}) & \leftarrow & \beta_2 = \mathcal{V}_{\Xi}(f; j+1, \frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2}, \frac{\varepsilon}{2}).
\end{array}$$

Then (4.10) reads

$$B^{\alpha_2} M_1 = M_2 B^{\alpha_1}.$$

So

$$\det B^{\alpha_2} = \frac{\det M_2}{\det M_1} \det B^{\alpha_1}.$$

Note that  $\frac{\det M_2}{\det M_1}$  is a determinant quotient computed in (4.11).

From (4.9), we get a relation between  $B_{12}$  and  $B_{33}$ :

$$\begin{aligned} \det \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} &= B_{11}B_{22} - B_{12}B_{21} \\ &= -\frac{1}{A_2 E^-} B_{33} (B_{33} A_2 E^+ - (E^- E^+ + A_1 A_2) B_{21}) . \end{aligned}$$

We can also compare (2, 1) entries of both sides in (4.10). Applying (4.9) and (4.12) to the both relations, we can finally write  $B_{21}$  in terms of  $B_{33}$  with a “big” help from computer algebra package.

$2 \times 2$  block on

$$\mathcal{V}_\Xi(f; j, \frac{1}{2}, \dots, \frac{1}{2}, \frac{\varepsilon}{2})$$

in terms of (3, 3)

$$\mathcal{V}_\Xi(f+1; j, \frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2}, \frac{\varepsilon}{2})$$

is:

$$\begin{pmatrix} \frac{4C_1 C_2}{(n-1)C_3 C_4} - 1 & \frac{-2(n-2)\Xi C_5 C_2}{(n-1)^2 C_3 C_4} \\ \frac{8n\Xi C_2}{C_3 C_4} & \frac{-4C_5 C_2}{(n-1)C_1 C_3 C_4} + \frac{C_6}{C_1} \end{pmatrix} \bullet Z(r; f+1, J, \Xi\varepsilon), \quad (5.14)$$

where

$$\begin{aligned} C_1 &= 2fn - 2f - 2n + 1 + n^2 + 2rn - 2r - 2\Xi J_a, \\ C_2 &= 2fr + \Xi J_a, \\ C_3 &= n - 1 + 2r, \\ C_4 &= (2f + 2r - \Xi + 2J_a)(2f + 2r + \Xi - 2J_a), \\ C_5 &= (n - 1 + 2J_a)(n - 1 - 2J_a), \text{ and} \\ C_6 &= 2fn - 2f - 2n + 1 + n^2 - 2rn + 2r + 2\Xi J_a. \end{aligned}$$

**Remark 2.** In particular, if  $r = \frac{1}{2}$  and (3, 3) entry

$$\sqrt{-1}f - \sqrt{-1}\Xi\varepsilon J$$

of the exchanged Rarita-Schwinger operator is put into the above formula, we recover the other  $2 \times 2$  entries

$$\begin{pmatrix} -\frac{n-2}{n}\sqrt{-1}\left(f + \frac{n+1}{n-1}\Xi\varepsilon J\right) & -\frac{2\sqrt{-1}\Xi}{n(n-1)}\left(\frac{(n-1)(n-2)}{4} - \frac{n-2}{n-1}J^2\right) \\ 2\sqrt{-1}\Xi & \sqrt{-1}f - \frac{n-3}{n-1}\sqrt{-1}\Xi\varepsilon J \end{pmatrix}.$$

## References

- [1] T. Branson. *Group representations arising from Lorentz conformal geometry.* J. Funct. Anal., **74** :199–291, (1987).
- [2] T. Branson. *Nonlinear phenomena in the spectral theory of geometric linear differential operators.* Proc. Symp. Pure Math., **59** :27–65, (1996).
- [3] T. Branson. *Stein-Weiss operators and ellipticity.* J. Funct. Anal., **151** :334–383, (1997).
- [4] T. Branson. *Spectra of self-gradients on spheres.* J. Lie Theory, **9** :491–506, (1999).
- [5] T. Branson, G. Ólafsson, and B. Ørsted. *Spectrum generating operators, and intertwining operators for representations induced from a maximal parabolic subgroups.* J. of Funct. Anal., **135** :163–205, (1996).
- [6] D. Hong. *Eigenvalues of Dirac and Rarita-Schwinger Operators.* Clifford Algebras and their Applications in Mathematical Physics, Birkhäuser, (2000).
- [7] D. Hong. *Spectra of higher spin operators.* Ph.D. Dissertation, University of Iowa, (2004).
- [8] Y. Kosmann. *Dérivées de Lie des spineurs.* Ann. Mat. Pura Appl., **91**:317–395, (1972).
- [9] B. Ørsted. *Conformally invariant differential equations and projective geometry.* J. Funct. Anal., **44**:1–23, (1981).